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ONE-DIMENSIONAL MAPS VIA COMPLEX ANALYSIS IN SEVERAL VARIABLES

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ABSTRACT

In this article we use elements from the theory of several complex variables to establish dynamical properties of certain one-dimensional maps (e.g. decay of correlations, central limit theorems, etc.).

Introduction

In this article we shall consider various properties of certain maps $f\colon I \to I$ of the interval. The types of dynamical behaviour we are interested in (e.g. exponential decay of correlations, central limit theorems, etc.) have been studied under a variety of different hypotheses. These properties are well-known to hold for uniformly expanding interval maps, and a natural problem is find weaker hypotheses under which they still hold true.

We shall take as our main hypotheses that

$$\begin{aligned} &\limsup_{n\to+\infty}\left(\int \frac{1}{|Df^n|}dx\right)^{1/n} < 1 \quad \text{and} \\ (0.1)\qquad &\limsup_{n\to+\infty}\left(\sup_{x\in \mathrm{Fix}(f^n)} \frac{1}{|Df^n(x)|}\right)^{1/n} < 1 \end{aligned}$$

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and we shall show that this leads very easily to dynamical results of this kind.

Related work on exponential decay has been done by both G. Keller, T. Nowicki and L.-S. Young under different hypotheses. In [8], Keller and Nowicki assumed the "Collet–Eckmann" hypothesis on the critical point c for a unimodal map $f: I \to I$, i.e.

$$\liminf_{n\to+\infty} \frac{1}{n} \log |Df^n(fc)| > 0. \tag{0.2}$$

In [17], Young studied families of unimodal maps and using estimates originating in the work of Benedicks and Carleson showed that for some parameter values (in fact, a set of positive Lebesgue measure) the unimodal map has an invariant measure with exponential decay of correlations.

Our approach is to work with the "resolvent" of the Perron–Frobenius operator. We follow the lead of Ruelle (in his treatment of zeta functions) by making use of the Bochner theorem for analytic functions of several variables. However, our application of this theorem requires a couple of new ingredients. Firstly, we need to consider **meromorphic** functions (more specifically, a class of meromorphic functions which when multiplied by the zeta function become analytic). Secondly, we need to study **Banach space valued functions** (for example, functions taking values in the space of L^1 functions).

1. Preliminaries on complex functions

In this section we shall be concerned with a general result involving analytic and meromorphic functions from an open domain $U \subset \mathbb{C}^{N+1}$, $N \geq 1$, to a complex Banach space B.

Definitions: A function F: $U \to B$ is **analytic** if for every point $a = (a_0, \ldots, a_N) \in U$ the function $F(z_0, \ldots, z_N)$ can be represented in a sufficiently small neighbourhood of a by the uniformly convergent series

$$F(z_0, \ldots, z_N) = \sum_{n_0=0}^{\infty} \cdots \sum_{n_N=0}^{\infty} b_{n_0 \cdots n_N} (z_0 - a_0)^{n_0} \cdots (z_N - a_N)^{n_N}$$

where $b_{n_0 \cdots n_N} \in B$. More generally, a function F: $U \to B$ is **meromorphic** if for every point $(a_0, \ldots, a_N) \in U$ there exists a neighbourhood $V \subseteq U$ of $(a_0, \ldots, a_N)$ and analytic functions F_1, F_2: $V \to \mathbb{C}$ with $F_2 \not\equiv 0$ and $F/V = F_1/F_2$. ∎

In the special case where B is merely the complex numbers $\mathbb{C}$ then these functions are the more familiar **complex valued functions**. For a nice account of the definitions and basic properties of general analytic and meromorphic functions we refer the reader to [6], chapter IX.

The main purpose of this section is to state a version of the Bochner Tube Theorem, a theorem which describes the analytic domain of functions of $N+1$ complex variables.

BOCHNER THEOREM: *Let $G(w_0, \ldots, w_N)$ be a function of $N+1$ complex variables*

$$(w_0, \ldots, w_N) \in \mathbb{C}^{N+1}$$

($N \geq 1$) whose analytic domain U includes the "tubes"

$$T_1 = \{(w_0, \ldots, w_N) \in \mathbb{C}^{N+1} \colon (\mathcal{R}(w_0), \ldots, \mathcal{R}(w_N)) \in K_1\},$$

$$T_2 = \{(w_0, \ldots, w_N) \in \mathbb{C}^{N+1} \colon (\mathcal{R}(w_0), \ldots, \mathcal{R}(w_N)) \in K_2\}$$

(where $\mathcal{R}(s)$ denotes the real part of s) for open neighbourhoods $K_1, K_2 \subset \mathbb{R}^{N+1}$ with $K_1 \cap K_2 \neq \emptyset$, then U must also include the larger "tube"

$$T = \{(w_0, \ldots, w_N) \in \mathbb{C}^{N+1} \colon (\mathcal{R}(w_0), \ldots, \mathcal{R}(w_N)) \in K\}$$

where

$$K = \{\alpha k_1 + (1-\alpha)k_2 \colon k_1 \in K_1, k_2 \in K_2 \text{ and } 0 \leq \alpha \leq 1\}$$

is the convex hull of the $K_1 \cup K_2$.

Remark: When $B = \mathbb{C}$ this is the usual Bochner Tube theorem, and a nice reference is [13] (cf. [4] for one generalisation to arbitrary Banach spaces). The original proof of Bochner constructs the analytic extension by using the Cauchy theorem to express the coefficients of a power series (relative to Legendre polynomials) and then estimating the radius of convergence by bounds on the coefficients [3]. These techniques apply equally well where the coefficients are in an arbitrary Banach space. Alternatively, Eric Bedford has shown us a derivation of the Banach space valued result from the usual $\mathbb{C}$ valued version. ∎

In the applications in this article, we shall want to apply the above theorem to certain special types of meromorphic function, as we explain below.

Given the tube

$$T_1 \cup T_2 = \{(w_0, \ldots, w_N): (\mathcal{R}(w_0), \ldots, \mathcal{R}(w_N)) \in K_1 \cup K_2 \subset \mathbb{R}^{N+1}\}$$

consider a meromorphic function $G(w_0, \ldots, w_{N+1})$. *Assume* that we can find an analytic (complex valued) function $k(w_0, \ldots, w_N)$ such that the product

$$(k.G)(w_0, \ldots, w_N) = k(w_0, \ldots, w_N).G(w_0, \ldots, w_N)$$

is again analytic. (This may not always be true for general functions, and the presentation of arbitrary meromorphic functions on a given domain, in this form, is related to the Poincaré problem [2], [9].)

We can apply the Bochner Theorem to deduce that if T is the tube associated to the convex hull K of $K_1 \cup K_2$ then there exists an analytic extension $k: T \to \mathbb{C}$. Similarly, the Bochner theorem gives that $k.G$ has an analytic extension to T. Finally, we deduce that $G = (k.G)/k$ has a meromorphic extension to the tube T.

2. One dimensional maps and the transfer operator

Consider a fixed map of the interval $f: I \to I$. We shall assume that we can partition I into intervals $I = I_1 \cup \cdots \cup I_k$ (with disjoint interiors) on each of which the function f is continuous and monotone and that the partition is **generating**, i.e. for any sequence $(i_n)_{n=0}^{\infty} \in \prod_{n=0}^{\infty}\{1, \ldots, k\}$ the intersection $\bigcap_{n=0}^{\infty} f^{-n} I_{i_n}$ consists of at most a single point. Furthermore, let us make the following assumption on the map $f: I \to I$.

Assumption on the map f: Assume that $f: I \to I$ has the following simple growth estimate

$$N := \limsup_{n\to\infty} (\#\mathrm{Fix}(f^n))^{1/n} < +\infty$$

where $\mathrm{Fix}(f^n)$ denotes the fixed points for f^n.

This condition will hold for a dense family of smooth maps on the interval (cf. [10], p. 508) and any polynomial map of degree $\deg(f) \geq 2$ (where $N \leq \deg(f)$). Consider a fixed (bounded) function $g: I \to \mathbb{R}^+$ which is continuous on each of the intervals I_i, $i = 1, \ldots, k$.

Growth functions: We can introduce the following growth functions:

$$\theta(g) = \limsup_{n\to\infty} \left(\sup_{x\in I} \prod_{i=0}^{n-1} g(f^i x) \right)^{1/n},$$

$$\beta(g) = \limsup_{n\to\infty} \left(\sup_{x\in \mathrm{Fix}(f^n)} \prod_{i=0}^{n-1} g(f^i x) \right)^{1/n}$$

and

$$\rho(g) = \limsup_{n\to\infty} \left(\sup_{x\in P_n} \prod_{i=0}^{n-1} g(f^i x) \right)^{1/n},$$

where we fix some point $x_0 \in I$ and denote $P_n(x_0) = \{x\colon f^n x = x_0\}$. It is immediate from these definitions that $\theta(g)$ is larger than either $\beta(g)$ or $\rho(g)$.

In the study of interval maps, two important tools are the following.

Definition:

(a) For complex variables $z, s \in \mathbb{C}$ we define a **zeta function** by

$$\zeta(z,s) = \exp\left(\sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{x\in \mathrm{Fix}(f^n)} \prod_{i=0}^{n-1} g^s(f^i x) \right)$$

(the domain of analytic convergence being non-trivial in all applications in this article.)

(b) For a complex number $s \in \mathbb{C}$ and a function $k\colon I \to \mathbb{C}$ we define the **Ruelle–Perron–Frobenius operator** (or **Transfer operator**) by

$$L_{g^s} k(x) = \sum_{fy=x} (g(y))^s\, k(y). \qquad \blacksquare$$

Remark: An important special case to keep in mind is the choice $g(x) = |Df(x)|$. We will then be interested in using the approach in this article to study the value $s = -1$, since this is intimately connected with the existence and behaviour of **absolutely continuous f-invariant measures**. We shall return to this point later. ∎

We still need to specify an appropriate space of functions preserved by the operator, which in turn depends on the properties of the transformation $f\colon I \to I$ and the weight function $g\colon I \to \mathbb{C}$. For any function $k\colon I \to \mathbb{C}$ we denote its supremum by $|k|_\infty = \sup_{x\in I} |k(x)|$ and its variation by

$$|k|_{BV} = \sup\left\{ \sum_{i=1}^{N} |k(x_i) - k(x_{i+1})| \colon x_1 < \cdots < x_N \right\},$$

and write $||k|| = |k|_\infty + |k|_{BV}$.

Assumption on g: We want to assume that

(a) g is real valued with $||g|| < +\infty$, and

(b) $||g^\alpha|| < +\infty$ for any $\alpha > 0$.

In particular, this implies that $|\theta(g^\alpha)| \leq N.|g^\alpha|_\infty < +\infty$. We shall henceforth assume this property without further comment.

Remark: Depending on the circumstances, we shall later want to assume one of the following additional assumptions: (i) $\beta(1/g) < +\infty$; or (ii) $\rho(1/g) < +\infty$. ∎

If we denote $B_{BV} = \{k\colon I \to \mathbb{C}\colon ||k|| < +\infty\}$, then the space B_{BV} can be shown to be a Banach space (with respect to the norm $||\ \ ||$), and L_g preserves the space B_{BV}. We let $\mathrm{sp}(L_g)$ denote the spectrum of this operator.

The **essential spectrum** $\mathrm{esp}(T)$ of an operator $T\colon B \to B$ on a Banach space B consists of those parts of the spectrum $\mathrm{sp}(T)$ of T which are left after we exclude eigenvalues $\lambda \in \mathrm{sp}(T)$ such that

(i) λ is an isolated eigenvalue;

(ii) $\mathrm{Range}(\lambda - T)$ is closed; and

(iii) $\bigcup_{k=1}^{\infty} \mathrm{Ker}(\lambda - T)^k$ is finite dimensional.

The **essential spectral radius** of $T\colon B \to B$ is given by

$$\rho = \sup\{|z|\colon z \in \mathrm{esp}(T)\}.$$

The following result is due to Baladi and Keller [1].

PROPOSITION 1:

(i) *The essential spectral radius of the operator $L_g\colon B_{BV} \to B_{BV}$ is bounded above by $\theta(g)$ (and therefore for $\mathcal{R}(s) > 0$ the essential spectral radius of $L_{g^s}\colon B_{BV} \to B_{BV}$ is bounded above by $\theta(g^{\mathcal{R}(s)})$),*

(ii) *For $\mathcal{R}(s) > 0$ the reciprocal of the zeta function $1/\zeta(z,s)$ is analytic for $1/|z| > \theta(g^{\mathcal{R}(s)})$ and the zeros in this domain occur at the values z for which $1/z \in \mathrm{sp}(L_{g^s})$.*

For $\mathcal{R}(s) > 0$, we can use Proposition 1 to describe the meromorphic domain of the resolvent of the operator $R(z,s) = (z - L_{g^s})^{-1}$ associated to L_{g^s}.

COROLLARY 1.1: *The operator $R(z,s)$ associated to L_{g^s}: $B_{BV} \to B_{BV}$ is meromorphic on the domain*

$$U = \{(z,s) \in \mathbb{C}^2 \colon |z| > \theta(g)^{\mathcal{R}(s)} \text{ and } \mathcal{R}(s) > 0\}.$$

Moreover, the singularities occur precisely at those values (z,s) such that $z \in \mathrm{sp}(L_{g^s})$ and we can write $R(z,s) = \zeta(1/z,s)N(z,s)$ where $N(z,s) \in L(B_{BV}, B_{BV})$ is analytic.

Proof: These conclusions follow from the series expansion of $R(z,s) = (z - L_{g^s})^{-1}$ being meromorphic on the domain U. We observe that $\theta(g^s) = \theta(g)^{\mathcal{R}(s)}$, which is immediate from the definitions, to deduce the domain of $R(z,s)$ from Proposition 1. ∎

For any fixed function $F \in B_{BV}$, the map $(z,s) \to R(z,s)F \in B_{BV}$ is meromorphic on the domain U. Moreover, the singularities may only occur at those values (z,s) such that $1/z \in \mathrm{sp}(L_{g^s})$ and we can write $R(z,s)F = \zeta(1/z,s)g(z,s)$, where $(z,s) \to g(z,s) = N(z,s)F \in B_{BV}$ is analytic on this domain.

Our reason for considering the image vector $R(z,s)F$ rather than the full operator $R(z,s)$ is easily explained. It is too restrictive to require that $R(z,s)$ is a linear operator on B_{BV} (for values $\mathcal{R}(s) \leq 0$), and it is more reasonable to consider the more general situation that for a fixed element $F \in B_{BV}$ the image $R(z,s)F$ is still in $L^1(I, dx)$ (for suitable values of z and s).

PROPOSITION 2: *Assume that g satisfies:*

(i) $\int L_{(1/g)} G dx = \int G dx, \forall G \in B_{BV}$;

(ii) *there exist constants $C > 0$, $\epsilon > 0$, and $0 < \gamma < 1$ such that*

$$\int \left| \prod_{i=0}^{n-1} \left(\frac{1}{g}\right)(f^i x) \right|^{\epsilon} dx \leq C\gamma^n, \quad \forall n \geq 0, \quad \text{and}$$

(iii) $\beta(1/g) < 1$.

Then, for any $F \in B_{BV}$ we have that $(z,s) \to R(z,s)F$ is meromorphic on the region defined by

(a) $|z| > \min\left(\gamma^{\left(\frac{|\mathcal{R}(s)|}{1+\epsilon}\right)}, \beta(1/g)^{|\mathcal{R}(s)|}\right)$; *and*

(b) $-(1+\epsilon) < \mathcal{R}(s) < 0$.

Proof: For convenience, we shall assume that $\gamma^{\left(\frac{1}{1+\epsilon}\right)} > \beta(1/g)$, the other case being similar.

We begin by considering the formal expansion

$$\begin{aligned} R(z,s)F &= \left(\frac{1}{z - L_{g^s}}\right) F \\ &= \frac{1}{z}\left(\frac{1}{1 - \frac{1}{z}L_{g^s}}\right) F \\ &= \sum_{n=0}^{\infty} a_n \frac{1}{z^{n+1}}, \end{aligned}$$

where $a_n = L_{g^s}^n F \in L^1(I, dx)$. Notice that we can write

$$\begin{aligned} \int |a_n| dx &= \int |L_{g^s}^n F| dx \\ &\le \int L^n_{\left(g^{\mathcal{R}(s)}\right)} |F| dx \\ &= \int L^n_{\frac{1}{g}} \left(\prod_{i=0}^{n-1}\left(\frac{1}{g}\right)(f^i x)\right)^{-(1+\mathcal{R}(s))} |F| dx \qquad (2.1) \\ &\le |F|_\infty \int L^n_{1/g} \left(\prod_{i=0}^{n-1}\left(\frac{1}{g}\right)(f^i x)\right)^{-(1+\mathcal{R}(s))} dx. \end{aligned}$$

If we assume $0 < d := -(1+\mathcal{R}(s)) < \epsilon$ we can bound

$$\begin{aligned} \int L^n_{1/g}\left(\prod_{i=0}^{n-1}\left(\frac{1}{g}\right)(f^i x)\right)^d dx &\le \left(\int \left(\left|\prod_{i=0}^{n-1}\frac{1}{g}(f^i x)\right|\right)^\epsilon dx\right)^{d/\epsilon} \qquad (2.2) \\ &\le C^{d/\epsilon}\left(\gamma^{d/\epsilon}\right)^n, \end{aligned}$$

using assumptions (i) and (ii) and the Hölder inequality. In particular, (2.1) and (2.2) together show that the map $(z, s) \to R(z, s)F$ taking values in the space $L^1(I, dx)$ is analytic on the domain $|z| > \gamma^{\frac{|\mathcal{R}(s)+1|}{\epsilon}}$ and $-(1+\epsilon) < \mathcal{R}(s) < -1$.

By the work of Ruelle we know that the complex function $\zeta(1/z, s)^{-1}$ is analytic on the domain $\mathcal{R}(s) < 0$ and $|z| > \beta(1/g)^{|\mathcal{R}(s)|}$ (cf. [15]).

If we write $z = e^w$ we have that $R(e^w, s)F$ and $\zeta(e^{-w}, s)^{-1}$ are analytic on the tube

$$T_1 = \{(w,s) \in \mathbb{C}^2 \colon (\mathcal{R}(w), \mathcal{R}(s)) \in K_1\}$$

where

$$K_1 = \{(x,y) \in \mathbb{R}^2 \colon x > \frac{(\log\gamma)}{\epsilon}|y+1|, x > |y| \log\beta\left(\frac{1}{g}\right) \text{ and } -(1+\epsilon) < y < -1\}.$$

In particular, notice that this set contains neighbourhoods arbitrarily close to $(x, y) = (\log\gamma, -(1+\epsilon))$.

We next consider a second domain. When $\mathcal{R}(s) > 0$ we can apply Corollary 1.1 to deduce that the function $(z, s) \to R(z, s)F \in L^1(I, dx)$ is meromorphic for $|z| > \theta(g)^{\mathcal{R}(s)}$, and takes the form $R(z, s)F = \zeta(1/z, s) \cdot N(z, s)F$. In particular, we see that both $\zeta(e^{-w}, s)^{-1}$ (by Proposition 1 (ii)) and $N(e^w, s)F$ are analytic on the domain

$$T_2 = \{(w, s) \in \mathbb{C}^2 \colon (\mathcal{R}(w), \mathcal{R}(s)) \in K_2\}$$

where

$$K_2 = \{(x, y) \in \mathbb{R}^2 \colon x > \log\theta(g) y \text{ and } y > 0\}.$$

In particular, K_2 contains neighbourhoods arbitrarily close to $(0, 0)$.

Finally, these two domains of analyticity can be connected by a tube with base K_3, say, corresponding to $-1 \le \mathcal{R}(s) \le 0$ and $|z| > |g|_\infty$, since

$$\begin{aligned}\int |a_n| dx &\le \int L^n_{g^{\mathcal{R}(s))}} |F| dx \\ &\le |F|_\infty \int \left(\prod_{i=0}^{n-1} \left(\frac{1}{|g|} \right)^{-(1+\mathcal{R}(s))} (f^i x) \right) dx \\ &\le |F|_\infty \left(|g|_\infty^{1+\mathcal{R}(s)} \right)^n\end{aligned}$$

and we are assuming that $|g|_\infty < +\infty$.

The domains T_1 and T_2 are both open tubes (and are connected via the tube corresponding to the domain K_3, introduced for this purpose). Using the comments on meromorphic functions after the statement of the Bochner Tube theorem, we can conclude that $N(e^w, s)F$ and $\zeta(e^{-w}, s)^{-1}$ are analytic, and thus the function $R(e^w, s)F = \zeta(e^{-w}, s)N(e^w, s)F$ is meromorphic, on a domain which includes the tube

$$T = \{(w, s) \in \mathbb{C}^2 \colon (\mathcal{R}(w), \mathcal{R}(s)) \in K\}$$

where

$$K = \left\{ (x, y) \in \mathbb{R}^2 \colon x > \left(\frac{\log\gamma}{1+\epsilon} \right) |y| \text{ and } -(1+\epsilon) < y < 0 \right\}.$$

This corresponds to the domain described by (a) and (b) in the statement of Proposition 2. This completes the proof. ∎

Remark: (1) The analytic extension of $\zeta(1/z,s)^{-1}$ by this method is due to Ruelle [15], and it was Ruelle's approach that inspired this proof.

(2) In the special case where we take $g = |Df|$ then we can use Fatou's lemma and the Hölder inequality to write

$$\int \liminf_{n\to+\infty}\left(\frac{1}{|Df^{(n)}|}\right)^{1/n} dx \le \liminf_{n\to+\infty}\left(\int \frac{1}{|Df^{(n)}|}dx\right)^{1/n} < 1.$$

In particular, we deduce that on a set of points x of positive Lebesgue measure, we have that

$$\limsup_{n\to+\infty}\left(|Df^{(n)}(x)|\right)^{1/n} > 1.$$

When the map f is unimodal and has negative Schwartzian derivative this condition implies the existence of an absolutely continuous invariant measure [7, Cor. 2(1)].

(3) We chose to formulate the hypotheses of Proposition 3 in terms of the integral with respect to Lesbegue measure. However, it is clear that the results are also equally valid using another finite probability measure μ (providing we then choose the function g so that $L^*_{1/g}\mu = \mu$).

(4) In [8] and [17] the spectrum of Ruelle–Perron–Frobenius operators were considered, albeit by very different methods. The assumptions made in those papers involved either the Collet–Eckmann conditions, or estimates of Carlson–Benedicks for quadratic maps. Our approach is completely different from either of these. ∎

THEOREM 1 (EXPONENTIAL DECAY OF CORRELATIONS): *Let μ be an absolutely continuous f-invariant measure, then we can write $d\mu = hdx$ where $h \in L^1(I,dx)$. For any $F \in B_{BV}$ and G with $|G.h|_\infty < +\infty$ we define the function*

$$\alpha(n) = \int G\circ f^n.Fd\mu - \left(\int Gd\mu\right)\left(\int Fd\mu\right) \quad \text{for } n \ge 0.$$

Assume that

(1) *there exist constants $C > 0$, $\epsilon > 0$, and $0 < \gamma < 1$ such that*

$$\int \left|\prod_{i=0}^{n-1}\left(\frac{1}{|Df^n(x)|}\right)\right|^{\epsilon} dx \le C\gamma^n, \quad \forall n \ge 0, \quad \textit{and}$$

(2) $\beta\left(\frac{1}{|Df|}\right) < 1$,

then for any

$$\eta > \max\left(\gamma^{\frac{1}{1+\epsilon}}, \beta\left(\frac{1}{|Df|}\right)\right)$$

there exist $\theta_1, \ldots, \theta_k \in \mathbb{C}$ *(of modulus at least* η*) and* $c_1, \ldots, c_k \in \mathbb{C}$ *such that*

$$\alpha(n) = \sum_{i=1}^{k} c_i \theta_i^n + O(\eta^n)$$

as $n \to +\infty$.

Proof: We shall denote $g = |Df|$ and observe that hypothesis (i) of Proposition 2 is immediate. We can assume without loss of generality that $\int F d\mu = 0$ and then we consider the following complex function:

$$\begin{aligned}
\rho(z) :&= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} \alpha(n) \\
&= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} \int L_{1/g}^n (G \circ f^n.F) dx \\
&= \int \left(\sum_{n=0}^{\infty} \frac{1}{z^{n+1}} L_{1/g}^n F\right) G d\mu \\
&= \int \left(\frac{1}{z - L_{1/g}} F\right) G d\mu \\
&= \int \left(\frac{1}{z - L_{1/g}} F\right) (G.h) dx.
\end{aligned}$$

By the previous proposition we know that this function is meromorphic for $|z| > \gamma^{1/(1+\epsilon)}$. In particular, this means that for any $\eta > \gamma^{(1/1+\epsilon)}$ we can find constants $\theta_1, \ldots, \theta_k \in \mathbb{C}$ which are poles for the resolvent $R(z, s)F \in L^1(I, dx)$. We can take the values $c_1, \ldots, c_k$ to be the corresponding residues of $\rho(z)$ such that

$$\alpha(n) = \sum_{i=1}^{n} c_i \theta_i^n + O(\eta^n)$$

as $n \to +\infty$. ∎

Under the same hypothesis we can also deduce certain central limit type theorems. (We can follow the treatment given in [11] and [12].)

We know that unity is an isolated maximal eigenvalue for $L_{1/g}$. We shall make the additional assumption that the eigenvalue is simple, with eigenvector

$L_{1/|Df|}h = h$. The associated absolutely continuous invariant measure takes the form $d\mu = hdx$. For each $n \in \mathbb{Z}^+$ we define a distribution on $\mathbb{R}$ by

$$G_n(t) = \mu \left\{ x \in I \colon \frac{\log |Df^{(n)}(x)| - n \int \log |Df| d\mu}{n^{1/2}} < t \right\}$$

and we let

$$N(t) = \frac{1}{(2\pi)^{1/2}\sigma} \int_{-\infty}^{t} e^{-y^2/2\sigma^2} dy$$

denote the Normal Distribution on $\mathbb{R}$ (with variance

$$\sigma^2 = \frac{d^2 \log \lambda(s)}{ds^2}\bigg|_{s=1},$$

where $\lambda(s)$ is the perturbation of the maximal eigenvalue associated to $L_{1/|Df|^s}$). The argument in [11] and [12] shows the following result.

THEOREM 2: *Let μ be a f-invariant probability measure for a map $f\colon I \to I$. Assume that*

(1) *there exist constants $C > 0$, $\epsilon > 0$, and $0 < \gamma < 1$ such that*

$$\int \left| \prod_{i=0}^{n-1} \left(\frac{1}{|Df^n(x)|} \right) \right|^{\epsilon} dx \le C\gamma^n, \quad \forall n \ge 0, \quad \textit{and}$$

(2) $\beta\left(\frac{1}{|Df|}\right) < 1$.

Then $G_n(t) = N(t) + O\left(1/n^{1/2}\right)$.

3. Distribution of pre-images of points and periodic points

Let us a fix a point $x_0 \in I$. We shall want to make the following standing assumption on pre-images throughout this section.

Assumption on pre-images: Assume that

$$\rho\left(\frac{1}{g}\right) = \limsup_{n\to\infty} \left(\sup_{x \in P_n} \left(\prod_{i=0}^{n-1} \left(\frac{1}{|g|} \right) (f^i x) \right) \right)^{1/n} < +\infty.$$

Remark: This hypothesis appears to be closely related to assumption (C1) in [5]. ∎

Definition: Given $F \in B_{BV}$, we can define (for variables $z, s \in \mathbb{C}$) complex functions

$$\eta_F(z) = \sum_{n=0}^{\infty} z^n \sum_{x \in P_n} \frac{F(x)}{\prod_{i=0}^{n-1} g(f^i x)}$$

and

$$\eta_F(z,s) = \sum_{n=1}^{\infty} z^n \sum_{x \in P_n} F(x) \left(\prod_{i=0}^{n-1} g(f^i x) \right)^{-s}$$

whenever they converge. ∎

Notice that $\eta_F(z) = \eta_F(z,1)$ and that we can write

$$\begin{aligned} \eta_F(z,s) &= \left(\sum_{n=0}^{\infty} z^n L_{1/g^s}^n \right) (F)(x_0) \\ &= \frac{1}{(1 - zL_{1/g^s})}(F)(x_0) \qquad (3.1) \\ &= \frac{1}{z} R\left(\frac{1}{z}, -s\right) F(x_0) \end{aligned}$$

Our main result on meromorphic domain of the function $\eta_F(z,s)$ is the following

LEMMA 1: *Let $\gamma = \max(\rho(1/g), \beta(1/g))$, then the complex function $\eta_F(z,s)$ is meromorphic whenever*

(a) $\gamma^{\mathcal{R}(s)}|z| < 1$,

(b) $\mathcal{R}(s) > 0$.

In particular, the complex function $\eta_F(z)$ is meromorphic on the domain $|z|\gamma < 1$.

Proof: Assume that M also gives an upper bound on the number of pre-images of $f\colon I \to I$, i.e. $\sup_{x \in I} \#\{f^{-1}(x)\} \leq M$. We first observe that the series presentation of $\eta_F(z,s)$ converges whenever

$$M\rho\left(\frac{1}{g}\right)^{\mathcal{R}(s)} |z| < 1 \quad \text{and} \quad \mathcal{R}(s) > 0$$

since, in particular, this implies that

$$|z| \limsup_{n\to\infty} \left(\sum_{x \in P_n(x_0)} \left(\prod_{i=0}^{n-1} \frac{1}{g}(f^i x) \right)^{\mathcal{R}(s)} \right)^{1/n} < 1.$$

In particular, this gives that $\eta_F(e^w, s)$ is analytic on the open tube

$$\log(M) + \mathcal{R}(s)\log\rho\left(\frac{1}{g}\right) + \mathcal{R}(w) < 0 \quad \text{and} \quad \mathcal{R}(s) > 0.$$

A similar argument (due to Ruelle, cf. [15]) shows that the complex function $\zeta(z,-s)^{-1}$ is analytic on the domain $|z|N\beta(1/g)^{\mathcal{R}(s)} < 1$ and $\mathcal{R}(s) > 0$, or equivalently, that $\zeta(e^w,-s)^{-1}$ is analytic on the domain

$$\log(N) + \mathcal{R}(s)\log\beta\left(\frac{1}{g}\right) + \mathcal{R}(w) < 0 \quad \text{and} \quad \mathcal{R}(s) > 0.$$

Providing $\mathcal{R}(s) < 0$, we can use the identity

$$\eta_F(z,s) = \frac{1}{z}R\left(\frac{1}{z},-s\right)F(x_0) = \frac{1}{z}\zeta(z,-s)N\left(\frac{1}{z},-s\right)F(x_0)$$

where both $N(1/z,-s)F$ and $z\zeta(z,-s)^{-1}$ are analytic on the domain $|z|\theta(g)^{\mathcal{R}(s)} < 1$ and $\mathcal{R}(s) < 0$, or equivalently $N(e^{-w},-s)F$ and $e^w\zeta(e^w,-s)^{-1}$ are analytic for

$$|\mathcal{R}(s)|\log\theta(g) + \mathcal{R}(w) < 0 \quad \text{and} \quad \mathcal{R}(s) < 0.$$

(In particular, we see that this domain contains elements arbitrarily close to $(\mathcal{R}(z), \mathcal{R}(s)) = (0,0)$.)

Applying the Bochner theorem to the two domains of analyticity for $\zeta(e^w,-s)^{-1}$ we can extend the region of analyticity to include the tube

$$\mathcal{R}(s)\log\beta\left(\frac{1}{g}\right) + \mathcal{R}(w) < 0 \quad \text{and} \quad \mathcal{R}(s) > 0,$$

or equivalently, $1/\zeta(z,s)$ is analytic for $\beta(1/g)^{\mathcal{R}(s)}|z| < 1$, cf. [15].

Similarly, applying the Bochner theorem to the two domains of analyticity for $N(e^{-w},-s)F$ we can improve the region of analyticity to include the tube

$$\mathcal{R}(s)\log\rho\left(\frac{1}{g}\right) + \mathcal{R}(w) < 0 \quad \text{and} \quad \mathcal{R}(s) > 0,$$

or equivalently, $N(z,s)F \in L^1(I,dx)$ is analytic for $\rho(1/g)^{\mathcal{R}(s)}|z| < 1$ and $\mathcal{R}(s) > 0$.

Together with the identity $\eta_F(z,s) = \frac{1}{z}\zeta(z,-s)N(1/z,-s)F$ this completes the proof of the lemma. ∎

Definition: We define $P_0 \in \mathbb{R}$ by

$$e^{P_0} = \limsup_{n\to+\infty} \left| \left(\sum_{f^n x = x_0} \prod_{i=0}^{n-1} \left(\frac{1}{g}\right)(f^i x) \right)^{1/n} \right| . \quad ∎$$

Assume that $e^{-P_0}\gamma < 1$ and $F > 0$, then by Lemma 1, $\eta_F(z)$ is meromorphic in a neighbourhood of $z = e^{-P_0}$. Moreover, by Pringsheim's theorem applied to $\eta_F(z)$ we see that $z = e^{-P_0}$ is a pole (of finite order $k \geq 1$, say).

We now come to a result on the distribution of pre-images.

THEOREM 3: *Assume that $e^{-P_0}\gamma < 1$ and $F \in B_{BV}$ then there exist*

(i) *constants $C_i \in \mathbb{C}$ $(i = 1, \ldots, N)$ and $\omega_i \in \mathbb{C}$, with $|\omega_i| = 1$ $(i = 1, \ldots, N)$,*

(ii) *positive integers k_i $(i = 1, \ldots, N)$,*

such that

$$\sum_{x\in P_n(x_0)} \frac{F(x)}{\left(\prod_{i=0}^{n-1} g(f^i x)\right)} \sim e^{P_0 n} \left(\sum_{i=1}^{N} C_i \omega_i^n \frac{(n+k_i)!}{n!} \right).$$

Proof: We can assume, without loss of generality, that $F > 0$. By the preceding comments, we know that $\eta_F(z)$ is meromorphic for $|z|\gamma < 1$, analytic for $|z|e^{P_0} < 1$ and has at least one pole on the circle $|z| = e^{-P_0}$. Assume that the poles on this circle occur at the values $z \in \mathbb{C}$ which satisfy $ze^{P_0}\omega_i = 1$, then we can write

$$\eta_F(z) = \sum_{i=1}^{N} \frac{R_i}{(1 - ze^{P_0}\omega_i)} + A(z)$$

where $A(z)$ is analytic on a neighbourhood of $\{z \in \mathbb{C}\colon |z|e^{P_0} < 1\}$, and the constants $R_i = R_i(F)$ are linear in F. Using the elementary expansions

$$\frac{1}{(1 - ze^{P_0}\omega_i)^{k_i}} = \frac{(-1)^{k_i}}{(k_i - 1)} \sum_{n=0}^{+\infty} z^n (e_0^P \omega_i)^{n+k_i} \frac{(n+k_i)!}{n!}$$

we can express $\eta_F(z) = \sum_{n=0}^{+\infty} a_n z^n + A(z)$ where

$$a_n = \left(\sum_{i=1}^{N} R_i \frac{(-1)^{k_i}}{(k_i - 1)} (e^P \omega_i)^{n+k_i} \frac{(n+k_i)!}{n!} \right).$$

By comparing this expansion with the definition of $\eta_F(z)$ we can deduce that

$$a_n \sim \sum_{x\in P_n(x_0)} \frac{F(x)}{\prod_{i=0}^{n-1} g(f^i x)}.$$

This is the required form. ∎

To approach the problem of the distribution of periodic points we need to first introduce a slight generalisation of our definition of a zeta function.

Definition: We define

$$\zeta(z,s,s_1) = \exp\left(\sum_{n=1}^{\infty} z^n/n \sum_{x\in \mathrm{Fix}(f^n)} \left(\prod_{i=0}^{n-1} g(f^i x)\right)^{-s} e^{-s_1 F^{(n)}(x)}\right)$$

with complex variables z, s, s_1 and a function $F\colon I \to \mathbb{C}$ of bounded variation, and we write $F^{(n)}(x) = \sum_{i=0}^{n-1} F(f^i x)$. ∎

Given such a function $F\colon I \to \mathbb{R}$ of bounded variation we define the xi function

$$\xi_F(z,s) = \sum_{n=1}^{\infty} z^n \sum_{x\in \mathrm{Fix}(f^n)} F(x) \left(\prod_{i=0}^{n-1} g(f^i x)\right)^{-s}$$

and we write $\xi_F(z) := \xi_F(z,1)$.

Observe that

$$\xi_F(z,s) = \frac{d}{ds_1}(\log \zeta(z,s,s_1))\,|_{s_1=0}$$

and that since

$$\limsup_{n\to\infty} \left| \sum_{x\in \mathrm{Fix}(f^n)} \left(\prod_{i=0}^{n-1} g(f^i x)\right)^{-s} \right|^{1/n} \le N\beta\left(\frac{1}{g}\right)^{\mathcal{R}(s)}$$

we know that $1/\zeta(z,s,s_1)$ is analytic on the domain

$$\left\{(z,s,s_1)\in \mathbb{C}^3\colon |z| N\beta\left(\frac{1}{g}\right)^{\mathcal{R}(s)} \beta\left(e^{-F}\right)^{\mathcal{R}(s_1)} < 1\right\}.$$

A second result on the analytic domain of $\zeta(z,s,s_1)$ is given by the following.

LEMMA 2 (Baladi–Keller): *The associated zeta functions $\zeta(z,s,s_1)$ satisfy that $1/\zeta(z,s,s_1)$ are analytic on the region*

$$|z|\theta(g)^{|\mathcal{R}(s)|}\theta(e^F)^{-|\mathcal{R}(s_1)|} < 1$$

providing $\mathcal{R}(s) < 0$; cf. [1].

Remark: We needed to assume that $\mathcal{R}(s) < 0$ in order that we know that $g^{-s}\colon I \to R$ is also a function of bounded variation. This is a necessary hypothesis for the Baladi–Keller approach. ∎

Applying the Bochner Theorem to these two domains for $\zeta(z, s, s_1)$ we can deduce the following result. Since the proof is completely analogous to that of Lemma 1 we prefer to omit it.

LEMMA 3: *$\zeta(z, s, 1)$ is analytic on a neighbourhood of the domain*

$$\left\{(s, z) \in \mathbb{C}^2\colon |z|\beta\left(\frac{1}{g}\right)^{\mathcal{R}(s)} < 1\right\},$$

and, moreover, $\xi_F(z, s)$ is meromorphic on the same domain. In particular, $\xi_F(z)$ is meromorphic on the domain $\{z \in \mathbb{C}\colon |z|\beta(1/g) < 1\}$.

The results above have immediate consequences for the distribution of periodic points.

Definition: We define $P_1 \in \mathbb{R}$ by

$$e^{P_1} = \limsup_{n \to +\infty} \left| \left(\sum_{x \in \mathrm{Fix}(f^n)} \prod_{i=0}^{n-1} \left(\frac{1}{g}\right)(f^i x) \right)^{1/n} \right|. \quad ∎$$

THEOREM 4: *There exist*

(i) *constants $C_i \in \mathbb{C}$ $(i = 1, \ldots, N)$ and $\omega_i \in \mathbb{C}$, with $|\omega_i| = 1$ $(i = 1, \ldots, N)$,*
(ii) *positive integers k_i $(i = 1, \ldots, N)$, and*
(iii) *a positive linear functional $L\colon B_{BV} \to \mathbb{C}$*

such that

$$\sum_{x \in \mathrm{Fix}(f^n)} \frac{F(x)}{\prod_{i=0}^{n-1} g(f^i x)} \sim e^{P_1 n} \left(L(F) \sum_{i=1}^{N} C_i \omega_i^n \frac{(n + k_i)!}{n!} \right).$$

The proof of Theorem 4 is completely analogous to that of Theorem 3, and therefore we shall omit the details.

4. The existence of eigenfunctions

We can now relate the analysis in the preceding section to the existence of certain eigenfunctions for the Ruelle–Perron–Frobenius operator.

PROPOSITION 3: *Assume that*

(i) *there exists a probability measure ρ such that $\int L_{(1/g)}G d\rho = \int G d\rho$, $\forall G \in B_{BV}$,*

(ii) *there exists $\epsilon > 0$ such that*

$$\int \left| \left(\prod_{i=0}^{n-1} \left(\frac{1}{g}\right)(f^i x) \right) \right|^{\epsilon} d\rho(x) \leq C\gamma^n, \qquad \text{for } n \geq 0$$

for some constants $0 < \gamma < 1$ and $C > 0$,

(iii) *$\beta(1/g) < 1$, and*

(iv) *$e^P \max\left(\beta(1/g), \gamma^{1/1+\epsilon}\right) < 1$ for some $P > 0$,*

then $z \to R(z,-1)1 \in L^1(I, d\rho)$ has a meromorphic extension to a neighbourhood of $z = e^{-P}$. If $z = e^{-P}$ is a pole, then there exists an eigenfunction $h \in L^1(I, d\rho)$ with eigenvalue e^{-P}.

Proof: By the proof of Proposition 2, we know that $z \to R(z,-1)1 \in L^1(I, d\rho)$ has a meromorphic extension to a neighbourhood of $z = e^{-P}$. We can then write that

$$R(z,-1)1 = \frac{h}{(z - e^{-P})^k} + A(z),$$

for some $k > 0$, where $z \to A(z) \in L^1(I, dx)$ is analytic in a neighbourhood of $z = e^{-P}$. In particular, we observe that

$$\begin{aligned} R(z,-1) &= \left(\frac{1}{z - L^n_{1/g}} \right) \\ &= \left(\sum_{n=0}^{\infty} \frac{1}{z^{n+1}} L^n_{1/g} \right)(1) \\ &= 1 + \frac{1}{z} L_{1/g} \left(\sum_{n=0}^{\infty} \frac{1}{z^{n+1}} L^n_{1/g} \right)(1) \\ &= 1 + \frac{1}{z} L_{1/g} \left(R(z,-1)(1) \right). \end{aligned}$$

Therefore, in a neighbourhood of the pole at $z = e^{-P}$ we have that

$$\frac{h}{(z - e^{-P})^k} + A(z) = 1 + \frac{1}{z} \frac{L_{1/g} h}{(z - e^{-P})^k} + \frac{1}{z} L_{1/g} A(z)$$

and comparing these singularities at $z = e^{-P}$ we see that $L_{1/g}h = e^{-P}h$. This gives the required eigenfunction. ∎

Application: Absolutely continuous invariant measures. We want to relate Proposition 3 to the existence of a measure ν on the interval I which is absolutely continuous with respect to Lebesgue measure. In particular, this means that we can find a function $h \in L^1(I, dx)$ such that $d\nu/dx = h$.

There is a well-known simple condition for the existence of such a function h. This is precisely that for the choice $g(x) = |Df|$ and $s = -1$ in the definition of the transfer operator we have that there exists a (positive) eigenfunction h for the eigenvalue 1 (i.e. $L_{1/|Df|}h = h$) [5]. If we have that for these choices

(i) $e^{-P} = 1$, and

(ii) there exists $\epsilon > 0$ such that

$$\int \left(\prod_{i=0}^{n-1} \frac{1}{|g(f^i x)|} \right)^{\epsilon} dx \leq C\gamma^n, \qquad \text{for } n \geq 0$$

for some choice $0 < \gamma < 1$,

then Proposition 3 (with $d\rho = dx$) guarantees the existence of an eigenfunction, and thus of an absolutely continuous invariant measure.

A very detailed account of the existence and ergodic properties of absolutely continuous invariant measures for interval maps is given in [7].

Appendix: C^k Markov maps

Consider the case where we make the following alternative assumptions on the map $f: I \to I$:

(a) it is Markov (with respect to the intervals of monotonicity); and

(b) it is uniformly expanding (i.e. $\inf_{x \in I} |Df(x)| \geq \lambda$, for some $\lambda > 1$);

(c) the function g and the map f are both C^r , for some $r \geq 1$, on the intervals of monotonicity.

For any C^r function k: $I \to \mathbb{C}$ we denote $|k|_i = \sup_{x \in I} |D^i f(x)|$, for $i = 1, \ldots, r$, and let $||f|| = |f|_\infty + |f|_1 + \cdots + |f|_r$. If we denote $B_{C^r} = \{f: I \to C: ||f|| < +\infty\}$, then the space B_{C^r} is a Banach space (with respect to the norm $||.||$), and L_g preserves B. For this operator it is known that the essential spectral radius of the operator L_g: $B_{C^r} \to B_{C^r}$ is bounded above by $\frac{1}{\lambda^r} \limsup_{n \to +\infty} |L^n_{|g|} 1|^{1/n}_\infty$ (cf. [16] or [14]). The following result applies to the function $(z, s) \to R(z, s) = 1/(z - L_{g^s})$ acting on B_{C^r}.

PROPOSITION A: *Assume that $F \in C^\infty$. There exists a constant $C > 0$ independent of r such that the map*

$$(z, s) \to R(z, s)F = \frac{1}{z - L_{g^s}} F \in B_{BV}$$

is meromorphic providing

$$|z| > \left(C^{1/2r} \frac{\lambda}{\theta(g)} \right)^{\mathcal{R}(s)} \quad \text{and} \quad 2r > \mathcal{R}(s) > 0.$$

In particular, the essential spectral radius of L_{g^s}: $B_{BV} \to B_{BV}$ is at most $\left(\frac{\lambda}{\theta(g)}\right)^{\mathcal{R}(s)}$.

Proof: We shall first establish that

$$(z, s) \to \frac{1}{z - L_{g^s}} F \in B_{BV}$$

is meromorphic (in a suitable sense) on two regions, and then use the Bochner Tube Theorem to complete the proof.

To describe the first region we recall that if $\mathcal{R}(s) \geq 2r$ then $|g(x)|^s = \left(g(x)^2\right)^{s/2}$ is C^{2r-1}, thus we begin with this restriction. It is easily deduced from the above comments on the essential spectral radius that the operator

$$R(z, s) = \frac{1}{z - L_{g^s}} \in L(B_{C^r}, B_{C^r})$$

is meromorphic whenever $|z| > e^{P(\mathcal{R}(s) . \log |g|)}/\lambda^{2r-1}$, and takes the form

$$R(z, s) = \zeta\left(\frac{1}{z}, s\right) N(z, s)$$

(where $\zeta(1/z, s)$ and $N(z, s)$ are analytic). Using the upper bound

$$e^{P(\mathcal{R}(s) . \log |g|^{\mathcal{R}(s)})} = \limsup_{n \to \infty} |L^n_{|g|} 1|^{1/n}_\infty \leq N\left(\theta(g)\right)^{\mathcal{R}(s)}$$

we know that $R(z, s) = \zeta(1/z, s) N(z, s)$ is meromorphic whenever

$$|z| > N\left(\theta(g)\right)^{\mathcal{R}(s)} / \lambda^{2r-1},$$

and thus under the stronger hypotheses

(i) $|z| > C\left(\frac{\theta(g)}{\lambda}\right)^{\mathcal{R}(s)}$ and
(ii) $\mathcal{R}(s) \in (2r, 2(r+1))$,
where $C > 0$ is a constant independent of r.

We can now consider a second region given by considering the operator acting on the larger space of functions of bounded variation. In particular, this implies that

$$R(z,s) = \frac{1}{z - L_{g^s}} \in L(B_{BV}, B_{BV})$$

is meromorphic in the region $|z| > \theta(g)^{\mathcal{R}(s)}$, provided $\mathcal{R}(s) > 0$, and takes the form

$$R(z,s) = \zeta(\frac{1}{z}, s)N(z,s),$$

where $N(z,s) \in L(B_{C^r}, B_{C^r})$ is analytic (by Proposition 1).

If we introduce the change of variable $z = e^w$, then these domains of analyticity for $\zeta(e^{-w}, s)$ and $N(e^w, s)F \in B_{BV}$ can be described by the tubes

$$T_1 = \{(w,s) \in \mathbb{C}^2 \colon (\mathcal{R}(w), \mathcal{R}(s)) \in K_1\}$$

where $K_1 = \bigcup_{r=1}^{\infty} K_1^r$ and

$$K_1^r = \left\{(x,y) \in \mathbb{R}^2 \colon x > \log C + \log\left(\frac{\theta(g)}{\lambda}\right) y \text{ and } y \in (2r, 2(r+1))\right\},$$

and

$$T_2 = \{(w,s) \in \mathbb{C}^2 \colon (\mathcal{R}(w), \mathcal{R}(s)) \in K_1\}$$

where

$$K_2 = \{(x,y) \in \mathbb{R}^2 \colon x > (\log\theta(g))y \text{ and } y > 0\}.$$

Observe that the domain K_1 has points arbitrarily close to $(\log(C) + \log\left(\frac{\theta(g)}{\lambda}\right) 2r, 2r)$ for each $r \geq 1$ and also that K_2 has points arbitrarily close to $(0,0) \in \mathbb{R}^2$.

We now see that, for any $r \geq 1$, the convex hull of $K_1 \cup K_2$ contains the set

$$K = \left\{(x,y) \in C^2 \colon x > \left(\frac{\log(C)}{2r} + \log\left(\frac{\theta(g)}{\lambda}\right)\right) y \text{ and } 2r > y > 0\right\}.$$

It follows from [15] that the complex function $\zeta(1/z, s)$ is analytic on the associated tube. By our observations on meromorphic functions in connection with the Bochner Tube theorem, we conclude that $R(z,s)F \in B_{BV}$ is meromorphic on the tube

$$T = \{(w,s) \in \mathbb{C}^2 \colon (\mathcal{R}(w), \mathcal{R}(s)) \in K\},$$

$(s,w) \in T$. This corresponds to the domain described in the statement of the Proposition on letting r tend to infinity. ∎

COROLLARY A.1 : *For $F \in B_{BV}$ the map*

$$z \to \frac{1}{z - L_g} F \in B_{BV}$$

is meromorphic on the domain $|z| > (\lambda/\theta(g))$.

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